Spring 2017 MATH5012

Real Analysis II

The Concentration Compactness Principle

Let $\mu_j, j \ge 1$, be a family of Radon measures on $\mathbb{R}^n$ satisfying $\mu_j(\mathbb{R}^n) = 1$ for all j. Here we are concerned with the convergence property of this family.

Theorem (Concentration Compactness Principle I) *There exists a subsequence of* $\{\mu_j\}$, $\{\mu_{j_k}\}$, *satisfying at least one of the following properties*:

(a) *For each* $\varepsilon > 0$, there is some r and k_0 such that

$$\sup_x \mu_{j_k}(B_r(x)) < \varepsilon \ , \quad k \ge k_0 \ .$$

(b) *There exists* $\{x_k\}$ *such that, for each* $\varepsilon > 0$, *one can find an* r *so that*

$$\mu_{j_k}(B_r(x_k)) > 1 - \varepsilon \ , \quad \forall k \ .$$

(c) *There exists some* $\lambda \in (0, 1)$ *so that, for each* ε *and* $r_0 > 0$, *one can find* $\{x_k\}$ *such that for each* $r' > r_0$, *there are Radon measures* μ_k^1, μ_k^2 *satisfying*

$$\mu_k^1 + \mu_k^2 \le \mu_{j_k} \ , \quad \text{supp}(\mu_k^1) \subset B_r(x_k) \ , \ \ \text{supp}(\mu_k^2) \in \mathbb{R}^n \setminus B_{r'}(x_k) \ ,$$

and

$$\limsup_{k\to\infty} \left(\left|\lambda - \mu_k^1(\mathbb{R}^n)\right| + \left|(1-\lambda) - \mu_k^2(\mathbb{R}^n)\right| \right) < \varepsilon \ .$$

The Levy function for μ_j is given by

$$Q_j(r) = \sup_x \mu_j(B_r(x)) \ , \quad r > 0 \ ,$$

and we set $Q_j(r) = 0$ for $r \leq 0$. Each $Q_j(r)$ is an increasing function on $(-\infty, \infty)$ bounded between 0 and 1. Let ν_j be the Radon measure taking Q_j as its distribution function. We have $\nu_j(-\infty, r) = Q_j(r)$ for all r. Since $\nu_j(\mathbb{R}) = 1$, we may pick a weakly convergent subsequence (theorem 1.41 in [EG]) and still denote it by ν_j, so $\nu_j \rightharpoonup \nu$ as $j \to \infty$. Let $Q^*(r)$ be the distribution function of ν. Since Q^* is increasing, its discontinuity set is at most countable. It is of full measure. Let C be its continuity set. For $r \in C$,

$$\lim_{j \to \infty} Q_j(r) = Q^*(r) \ , \tag{1}$$

see theorem 1.40 (iii). We let

$$\lambda = \lim_{r \to \infty} Q^*(r) \in [0, 1] \ .$$

Case (a) $\lambda = 0$. For each $r > 0$, we fix some $r_1 > r, r_1 \in C$, such that $Q^*(r_1) < \varepsilon$. Then there is some j_0 such that $Q_j(r_1) < \varepsilon$ for all $j \geq j_0$. Therefore,

$$Q_j(r) \leq Q_j(r_1) < \varepsilon \ , \quad \forall j \geq j_0 \ ,$$

and (a) follows.

Case (b) $\lambda = 1$. We first determine $\{x_j\}$. Indeed, from $\lim_{r \to \infty} Q^*(r) = 1$ we fix some $r_0 \in C$ such that $Q^*(r_0) > 1/2$. By (1) $Q_j(r_0) > 1/2$ for all $j \geq j_0$ for some j_0. Enlarge r_0 if nec, we may assume indeed it holds for all $j \geq 1$. Then we can pick x_j such that $\mu_j(B_{r_0}(x_j)) > 1/2$ for all $j \geq 1$. Now $\{x_j\}$ has been picked. Next, for $\varepsilon \in (0, 1/2)$, we pick r_1 and $\{y_j\}$ by a similar reasoning so that $\mu_j(B_{r_1}(y_j)) > 1 - \varepsilon$ for all j. In view of

$$\mu_j(B_{r_0}(x_j)) + \mu_j(B_{r_1}(y_j)) > \frac{1}{2} + 1 - \varepsilon > 1 \ ,$$

$B_{r_1}(y_j)$ and $B_{r_0}(x_j)$ must intersect, so

$$B_{r_1}(y_j) \subset B_r(x_j) \ , \quad r = r_0 + 2r_1 \ .$$

It follows that

$$\mu(B_r(x_j) \geq \mu_j(B_{r_1}(y_j)) > 1 - \varepsilon \ , \quad \forall j \ ,$$

done.

Case (c) $\lambda \in (0,1)$. As $\lambda = \lim_{r\to\infty} Q^*(r)$, for $\varepsilon > 0$, we can pick an $r_0 \in C$ such that $Q^*(r_0) > \lambda - \varepsilon/2$. Arguing as before we can find $\{x_j\}$ such that $\mu_j(B_{r_0}(x_j)) > \lambda - \varepsilon/2$, for all j. On the other hand, using the fact $\mu(B_r(x_j)) \to 1$ as $r \to \infty$, for each j there is some $r_j > r_0$, $r_j \to \infty$ as $j \to \infty$, satisfying $\mu_j(B_{r_j}(x_j)) < \lambda + \varepsilon/2$, for all j. We set

$$\mu_j^1(E) = \mu_j(E \cap B_{r_0}(x_j)) \ , \qquad \text{and } \mu_j^2(E) = \mu_j(E \cap \mathbb{R}^n \setminus B_{r_j}(x_j)) \ .$$

Then for each $r' > r_0$, the support of μ_j^2 is contained outside the ball $B_{r'}(x_j)$ for all sufficiently large j. We have

$$\lambda - \mu_j^1(\mathbb{R}^n) = \lambda - \mu_j(B_{r_0}(x_j)) < \lambda - (\lambda - \varepsilon/2) = \varepsilon/2 \ .$$

Also

$$\lambda - \mu_j(B_{r_0}(x_j)) \geq \lambda - \mu_j(B_{r_j}(x_j)) \geq \lambda - (\lambda + \varepsilon/2) = -\varepsilon/2 \ .$$

Next,

$$(1-\lambda) - \mu_j^2(\mathbb{R}^n) = (1-\lambda) - \mu_j^2(\mathbb{R}^n \setminus B_{r_j}(x_j)) = (1-\lambda) - (1 - \mu_j(B_{r_j}(x_j))) < \lambda + \varepsilon/2 - \lambda = \varepsilon/2 \ ,$$

and

$$(1-\lambda) - \mu_j^2(\mathbb{R}^n \setminus B_{r_j}(x_j)) = -\lambda + \mu_j(B_{r_j}(x_j)) \geq -\lambda + \mu_j(B_{r_0}(x_j)) > -\varepsilon/2 \ .$$

We have completed the proof of the theorem.

The concentration-compactness principles were introduced by PL Lions in ninety eighties in a series of papers. This version is taken from M Struwe "Variational Methods" with some modifications. The reader may find many applications of these principles in the original and subsequent papers.

Very roughly speaking, in the calculus of variations one deals with the minimization of certain functionals of the form

$$J(f) = \int_{\mathbb{R}^n} F(x, f(x), \nabla f(x)) d\mathcal{L}^n(x) \ ,$$

subject to some constraints such as

$$\int_{\mathbb{R}^n} |f(x)| d\mathcal{L}^n(x) = 1 \ .$$

Under very general assumption on F, $J(f)$ has a finite lower bound for all functions f under consideration. Hence we can find $\{f_j\}$ such that

$$J(f_j) \to \inf \{J(f) : \ f \text{ admissible } \} > -\infty \ , \qquad \text{as } j \to \infty \ .$$

The key issue to establish the convergence of $\{f_j\}$ to a minimizer. A first move is to view

$$\mu_j(E) = \int_E |f_j(x)| d\mathcal{L}^n(x)$$

as a sequence of probability measures. By the first concentration-compactness principle, we can extract a subsequence which is again a minimizing sequence fulfilling one of the three possibilities. In case we can exclude the first and the third cases, μ_j would converge weakly to some probability measure μ. If we could further show that $\mu << \mathcal{L}^n$, then $\mu = f\mathcal{L}^n$ and f_j would converge weakly to some L^1-function f and this f is our candidate for the minimizer. The exclusion of cases (a) and (b) in the theorem depends on the function F which involves the derivative of f. Other concentration-compactness principles come into play to achieve this goal.